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Fractals: The Geometry of Complex Shapes

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ABSTRACT

This paper reviews the fractal approaches and its applications to the characterization of complex objects. These concepts are universally used in diverse branches ranging from particle physics to cosmology from languishes to finance and so on. Inadequacies of Euclidean geometry are discussed.

Keywords: Fractals, Complex systems.

1. INTRODUCTION

Universe is made up of many different structures arranged in a fairly well-defined hierarchy ranging from quarks to the galaxies, their clusters and super clusters etc**.** The fractals geometry has found lots of applications in various fields of science and engineering. Generally geometry deals with two types of entities, objects and spaces. Points, lines, spheres rectangular parallelepipeds are the objects that live in spaces. A zero dimensional point that lie on the one dimensional line or on a two dimensional plane. A point which is zero dimensional object, lies on a one dimensional space or a two dimensional space. Likewise a line can lie in a two dimensional space or a curve can lie in a three dimensional space. In other words, we can say that objects lie in spaces of dimension either greater than or equal to dimension of objects.

The laws of Pythagoras and Euclid, devises the concept of idealization that are spheres, triangles, rectangular parallelepipeds. Such idealized objects are studied in flat spaces. In such spaces the minimum distance between two points is given by Pythagoras law. There can be other types of spaces, like curved spaces. The concept of curved space have found profound application that in the theory of gravitation, where Pythagorean approach can not be applied. This shows the difference between two types of geometry, a flat space and the curved space but the object still remains a triangle of idealized type.

After B. B. Mandelbrot[1-6], the science is beginning to recognize the thing that are far from idealization, things that are crooked, bent and require different geometry and fractal geometry essentially concern with the things that we see around us. The natural objects are continuous but not

differentiable geometric objects. The next major difference illustrated by Beniot Mendalbrot who invented this fractal geometry was that if we measure the coastline of England then its measure depends on the yard stick. As the yard stick measurement goes to zero,the measured length of coast line goes to Infinitive. Similar property occurs for natural surface also. Lungs, that absorb as much oxygen as possible from the air as a result for this purpose it should have maximum surface area to expose to air but it has finite volume, a small possible volume because it has to be accumulated to the reef cage and immediately leads to the conclusion that surface of lungs must of the same character that is surface of infinite area yet enclosing a finite volume.

There are the aspects in which the natural geometrical objects differ from the ideal objects. They are continues but non differentiable. These kinds of objects need a different kind of characterization. We need different technique that is known as dimension. The concept of the dimension of object and the embedding space are two different things. The dimension of the embedding space is generally given by degree of freedom. Since degrees of freedom are integers therefore the dimension of embedding space must be an integer but that does not means that dimension of object should also be an integer because the objects lives in the space and its dimension is understood by how it fills the space.

2. NOTION OF DIMENSION

Consider a curve. Enclose it in a cube of size L. Next, divide the cube into equal cubes of size L/k. the numbers of cube is k^3, however, the curve need not pass through each of these cubes, Let the number of cubes through which the curve passes be N_1. This is the first generation. Now get the second generation characterization by dividing each of these small cubes is now $(2k)^3$. Some of these cubes are empty. We can again compute the number of non empty cubes. In this we get larger and larger numbers of smaller and smaller cubes. In general we find that the total number of non-empty cubes denoted, by N is proportional to l^D, where l is the linear size of the smallest elemental cube. So given a distribution of points, we evaluate dimension D by computing,

$$D = \lim_{l\to 0} \frac{d \ln(N)}{d \ln(l)} \qquad (3)$$

The exponent D will turn out to be one. Similarly if we do this to a two dimensional smooth surface embedded in 3 dimensional and perform the above procedure then the exponent will turn out to be 2. So we generalize the concept of dimension and state that this exponent computed in the above way will be called dimension of the object. The interesting thing is the case of fractals, this exponent need not be an integer. It can be a fraction. This is the origin of the nomenclature of such objects.

3. SELF-SIMILARITY DIMENSION

Another useful definition of dimension is the self-similarity dimension. At each stage in the construction of the Koch curve, the previous figure is reduced by a factor of three and four such reduced figures make the next figure. The limiting figure

that will be obtained if the process is continued through infinite stages will be such that if reduced by a factor of three, four such reduced figures will reproduce the original figure. Self-similarity dimension D_s depends on these two numbers. If the figure is reduced by the factor r, and n such reduced figures reproduce the original figure, we say that the figure is self-similar with self-similar dimension

$$D_s = \log(n) / \log(r) \quad (4)$$

Thus the self similarity dimension for the Koch curve is

$$D_s = \log(4) / \log(3) \quad (5)$$

Similarly for the case of a cantor set, the similarity dimension turns out to be ln (2) / ln (3).

Consider an example of self-similar object i.e. cantor set, choose a particular portion on line and try to generate fractal, which is of the distance between 0 and 1. So we are attempting to generate the fractal. In first step divide line into three equal parts and then eliminate the middle one third. In the next iteration we will continue the process, again remove the one third and on continuing this process up to infinite terms we will be essentially left with a collection of dots on the real line each of segments will sink to zero size, and we will be left with the sprinkles of dots.

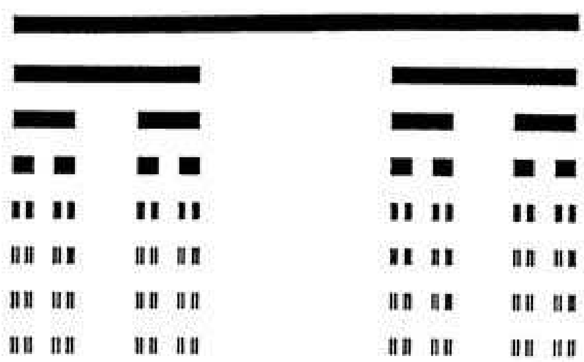

Fig:10 Representation of cantor set

Now if we have to find the dimension of this object, then cover the object through boxes and then count the number of boxes necessary to cover this object. So here we need only to create boxes within the range 0 and 1, and subdivide it into 10 numbers of boxes like 0 to 0.1, 0.1 to 0.2, 0.2 to 0.3…………..0.9 to 10. And then count and further subdivide it, then let us see the mathematical representation of the number of boxes needed to cover the object.

We have N = 2 and $\in = 1/3$.

Putting these values in equation $D = \frac{\ln N(\epsilon)}{\ln 1/\epsilon}$.

After first iteration so we get

$D = \frac{\ln 2}{\ln 3}$ = fractional number lying between 0 and 1.

After second iteration we get

N = 4 and $\in$ = 1/9,
We get D = $\frac{\ln 4}{\ln 9} = \frac{\ln 2}{\ln 3}$.

We see here that after first and second iteration the dimension measured remains the same, now if continue this procedure up to infinitive number of iteration we will be left with the collections of dot, thus this whole procedure is known as cantor set its dimension should also be same i.e. ln 2/ln 3, this is fractional number here we conclude that this object is fractal. Therefore a fractal which has embedding space of one dimension, there fractal dimension will be between 0 and 1, which is two in case of cantor set and is very important for the study of non linear dynamics. Now let us consider another condition where the embedding space is two dimensional. Consider a square in this also apply middle one third removing procedure, and to obtain its dimension we can mathematically represent it through following iterations.

After first iteration if N = 8 and $\in$ = 1/3 then,

D = $\frac{\ln 8}{\ln 3}$ = number between 1 and 2.

After second iteration if N = 64 and $\in$ = 1/9 then,

D = $\frac{\ln 64}{\ln 9} = \frac{\ln 8}{\ln 3}$.

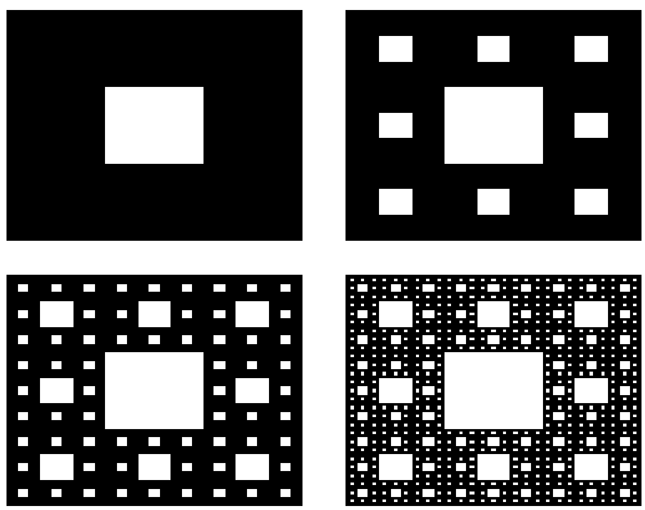

Fig:11 Construction of the Sierpinski square.

The initiator is a Square and the generator (shown on the left-hand side) is made of N = 8 Squares. They are obtained by contractions of ratio r = 1/3. The right – hand side of the figure shows the fourth construction stage.

So after performing number of iteration in this two dimensional space we will be left with only some parts filled and rest of the parts removed empty and that will be a collection of points in two dimensional space and this set is called as sierpinski

square, which is the another important example of self-similar object. So we conclude that on reducing $\in$ we obtain the same dimension and therefore the dimension is really invariant while $\in \rightarrow 0$. There is another variant of the Sierpinsk construction which is a Sierpinsk triangle, so in this case we construct a triangle and here also we will apply the removal of middle one third part thus the triangle get divided into four equal parts and in this procedure we remove the middle part of that triangle, so now we are left with three triangles so this was the first iteration in the same way in the next iteration we will take each of there chunks and remove further, so in every iterate we keep on removing each parts and after repeating these steps again and again we ultimately get the collections of dots.

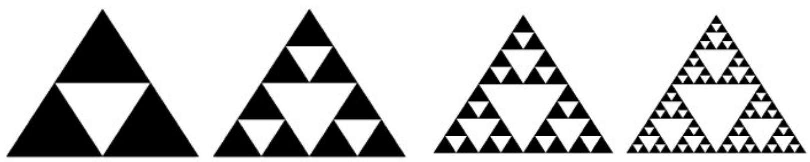

Fig:12 Construction of the triangular Sierpinski gasket, whose initiator is filled with triangle. The generator eliminates a central triangle as shown.

Now to calculate its dimension, we subdivide the embedding space into the grid. And cover the object through boxes and count the number of boxes required to cover the object. Let us represent this mathematically through number of iterations.

In first iteration N = 3, ε = ½ so we get,

D = $\frac{\ln 3}{\ln 2}$.

In second iteration N = 9, $\varepsilon = 3/4$ so we get,

D = $\frac{\ln 9}{\ln 4} = \frac{\ln 3}{\ln 2}$.

So here also we see that on subdividing the embedding space and making ε smaller we get the same value of dimension. Here we notice that the dimension is between 0 and 1thus we have created an object which is a fractal. To understand such structures, we need to take a more quantitative look at these. In the next section, we describe the algorithm to generate a particular type of fractal, the Koch curve. This example will help us appreciate the unfamiliar behavior of these structures.

4. CORRELATION DIMENSION

The box counting dimension counts every box that has a point of the object inside the box: it does not care how many points of the object there are in the box. For a mathematical fractal, quite obviously, if a box contains one point, it contains infinite number of points, because the scaling behavior continues indefinitely. In physical applications, however, there is usually a lower and upper cut off beyond which the fractal-scaling behavior does not hold. Moreover, instead of all the points of the object, one has access to only a relatively small sample. The concept of correlation

dimension is useful for handling such situations. The correlation dimension describes the scaling of N(r) with r, if N(r) is proportional to r^D, D is called the correlation dimension.

Self-similar structures must always be fractal is a false assumption. For example, line segments, or squares, or cubes are all self-similar but not fractal, in these cases the reduction factor can be arbitrary. But for fractals the reduction factor are characteristic. For example, the Koch curve can be reduced only by factors of 1 / 3^n to obtain self-similarity.

However, for all self-similar structures, whether fractal or not, there is always a relation between the scaling factor and the number of scaled down pieces into which the structure is divided. Consider a line segment of length, say 8 cm. If they are scaled down by a factor 1 / 2. We will get a line segment of length 4cm. Two such scaled down line segments put together produce the original line segment of length 8 cm. more generally, if a line is scaled down by a factor 1 / n then n such scaled down pieces are required to obtain the original line. The self-similarity dimension will be.

$$D_s = \frac{\ln(n)}{\ln(\frac{1}{1/n})} = 1 \quad (5)$$

Consider a square of side, say 8 cm. if it is scaled down by a factor of 1 / 2. We will get a square of side 4 cm four such scaled down squares put together produce the original square of side 8 cm. if a square is scaled down by a factor 1 / n then n^2 such scaled down squares will be required to obtain the original square. The self-similarity dimension will be.

$$D_s = \frac{\ln(n^2)}{\ln(\frac{1}{1/n})} = 2 \quad (6)$$

Consider a cuse of side, say 8 cm. if it is scaled down by a factor of 1 / 2. We will get a cube of side 4 cm. Eight such scaled down squares put together produce the original square of side 8 cm. if a square is scaled down cubes of side 8 cm. More generally a, if a cube is scaled down by a factor 1 / n then n^3 such scaled down cubes will be required to obtain the original cube. The self-similarity dimension will be.

$$D_s = \frac{\ln(n^3)}{\ln(\frac{1}{1}/n)} = 3 \quad (7)$$

If a Koch curve is scaled down by a factor 3^{-n}, then 4^n such scaled down Koch curves will be required to obtain the original Koch curve. The self-similarity dimension will be

$$D_s = \frac{\ln(4^n)}{\ln(\frac{1}{3^{-n}})} = \frac{\ln(4)}{\ln(3)} \quad (8)$$

The above examples indicates that the silf-similarity dimension is a generalization of our ordinary understanding of dimension in as much as 'nice' objects such as line, square and cube have self-similarity dimension equal to their 'ordinary' dimension.

There is another rumor that a bounded curve having infinite length must be a fractal is also false. The assumption that fractal dimension must be a fractional is also a false rumor.

It is a common mistake to think of fractals as objects having non-integer dimensions. Devil's sraircase and peano curve are examples of fractal curves that have integer dimensions.

5. RANDOM FRACTALS

So far we have considered what is called deterministic fractals. Such fractals do not occur in Nature. Nature presents what are called statistical fractal is obtained if we modify the construction of the Koch curve. When we place a line segment by the generator shown in figure, we let this generator lie on either side of the line with equal probability. A stage 5 construction of this curve is shown in figure.

6. CONCLUSION

The fractal nature of universe is still a matter of intense debate. We now have very deep surveys. But these are not yet deep enough to make a definitive statement. Though some observations, indicates that there is a transition to homogeneity at a scale of about 100 Mpc. But it appears that although we have at our disposal deep surveys, they are not even deep enough to give decisive condition about the transition. To make a decisive statement we still need deep observations. Specially, the observation are needed on the theoretical side, the likelihood of cosmic evolution being governed by a chaotic dynamical system indicates that the Universe could have a fractal structure.

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